Groebner bases under composition and multivariate matrix factorization

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Contents of the talk

In this talk, I will introduce some of my work related to Groebner bases.

- I.Behavior of a Groebner basis under composition — New Results;
- II. Multivariate polynomial matrix factorization — New progress.

PART(I)

Groebner bases under composition

Groebner bases under composition

Hoon Hong initiated the study of Greobner bases under composition

Groebner bases under composition

Early papers:

- (i). H.Hong, *Groebner bases under composition II*, ISSAC, 2006.
- (ii) H.Hong, *Groebner basis under composition I*, JSC (1998), 25, 647-662.
- (iii) Gutierrez , J. Miguel, R, *Reduced Groebner basis under composition*, JSC(1998), 26, 433-444.

Basic research Problem

Let us consider the polynomial ring in n variables:K[x]=k[x_1,…x_n], and a given term ordering > .

The basic problem can be described as follows:

For an endomorphism U of k[x], and a finite subset G of k[x], how can we compute a Groebner basis of U(G) under the term oredering > by means of a Groebner basis of G and U?

Hong's Theorem

For any Grobner basis G of k[X], U(G) is a Groebner basis if and only if

- (a) for any terms p and q, p<q implies lt(U(p))<lt(U(q));
- (b) lt(U) is a permuted powering , that is, every component of U has a power of one variable as the leading term, where U=(u1,…un): xj→uj. and lt(U)=(lt(u1),…,lt(un)).

Result of Gutirrez etc.

Gutirrez etc. have proved a result on reduced case. Their result is:

For any reduced Groebner basis G, U(G) is a reduced Groebner basis if and only if

- (a) for any terms p, q, p<q implies lt(U(p))<lt(U(q)).
- (b) every component of U is a polynomial in one variable, and different polynomial involves in different variable.

A generalization

First, We establish a more general framework:

Let U be a homomorphism from k[x]= K[x1,…xn] to k[y]=k[y1,…ym] s.t. n<m or n=m.

Given two term orderings $>_1$ on k[x] and $>_2$ on K[y]. In this case uj is in k[y].

A generalization

Under this new framework, Hong's theorem can be **generalized** as follows:

For any Groebner basis G with respect to $<_1$, U(G) is a Groebner basis with respect to $<_2$ if and only if

- (1) for any terms p, q, $p <_1 q$ implies that $lt(U(p)) <_2 lt((U(q))$;
- (2) lt(ui) and lt(uj) are pairwise coprime for different i and j.

A generalization

But in reduced case, it seems to lack a sufficient and necessary condition in this generalization case.

Details can be found in:

Remarks on Groebner basis for ideals under composition, ISSAC, 2001.

Check the term ordering conditions

From the above theorems, we need to solve **the term ordering compatible problem** in order to apply these theorems:

Find an efficient algorithm to check if for any terms p and q, $p <_1 q$ implies $lt(U(p)) <_2 lt(U(q))$?(Hong)

Continue

- Let T be the matrix corresponding to exponent vectors of leading terms of uj, where U=(u1,…un)
- Let term ordering $>_1$ be represented by a nxn matrix A , and $>_2$ be represented by matrix B.
- Thus our method is using rational elementary transformation to A and TB simultaneously to obtain some standard form in some sense, then check if these standard form are the “same” which is up to a positive number multiple.

Continue

So called elementary rational transformation for the real matrices we mean:

- I. Multiplying a row of a matrix by a non-zero rational number;
- II. Interchanging any two rows;
- III. Adding a rational multiple of one row to another row

See JSC(2003), vol.35.

Homogeneous case

An interesting problem is:

Under what conditions that for any homogeneous groebner bases G, U(G) is homogeneous greobner bases?

Journal of Algebra, computational algebra section, In press.

Homogeneous case

We have provided a complete answer:

For any homogeneous Greobner bases G, U(G) is a homogeneous Groebner bases if and only if

- (I) for any terms p,q with deg(p)=deg(q), p<q implies lt(u(p))<lt(u(q));
- (ii) lt(U) is a permuted powering, and every ui has the same degree.

Further results

The problems can be further extended more general case. Let L be an arbitrary grading on k[x], we may ask:

Under what conditions that for any L -homogeneous groebner bases G, U(G) is Groebner bases?

An answer

For any L- homogeneous Gb G, U(G) is GB if and only if

- (I) for any terms p>q, L(p)=L(q) implies lt(U(p))>lt(U(q));
- (II) lt(U) is a permuted powering;

Remarks

- This new result unifies all the previous results. Because if L(xi)=0 for every I, we get Hong's theorem; if L(xi)=1 for every I, we get the usual homogeneous result.

- The proof is slightly hard; submitted to Journal of Algebra, CA section.

Summary

- From above descriptions, we see that only universal cases are considered.
- For the basic problem: Given a Groebner basis G, and a homomorphism U, How can we compute the Groebner basis of U(G) by means of G and U?
- We have not solved the basic problem! It seems to have some difficulties.

Summary

- There are a lot of related work in Resultant, subresultant, Sagbi bases, non-commutative Groebner bases etc.
- See related papers.

PART(II)

Multivariate polynomial matrix factorization

Basic Problem

Given a nD polynomial matrix F with full row rank of size (l, m).

Let d be the gcd of all the l-minors, f|d, whether or not there exists square matrix G and matrix F1,such that

F=GxF1 with det(G)=f?

The History

- When n=1, this is solved completely using Gauss elimination, and play an essential role in linear systems and control theory, convolutional codes.
- When n=2, this is solved using pseudodivison by N. K. Bose and others, used in 2-D systems and signal processing.
- When n>=3, this is thought as very hard! Because previous methods can not be generalized to this case.

Motivation for research

Why we concern about this problem?

- (i). Generalize linear system theory to multidimensional systems theory since 1976.
- (ii). Many problems in multidimensional systems and signal processing can be formulated in multivariate matrix problems.
- (iii). May be useful in multidimensional convolutional codes.
- (iv). Possibly other applications.

Reduced minors

In order to introduce Lin-Bose Problem, we give a basic definition as follows:

Let F be a full row rank matrix of size (l, m), let a1,…,ak be all the l-minors of F, d be the g.c.d of a1,…,ak, and let bj such that aj=dxbj . b1,…,bk is said to be the **reduced minors** of F.

Some research papers

There are many papers concerning this problems, we just mention:

- C. Charoenlarpnopparut, N. K.Bose, Multidimensional FIR filter Bank Design using Groebner basis, IEEE Trans. Circuits Systems II Vol 46, 1999, pp1475-1486.
- Z. Lin, Notes on nD polynomial matrix factorization, Multidimensional system and Signal Processing, Vol. 10, 1999, 379-393.

Research papers

- Z. Lin, Further results on nD polynomial matrix factorization, Multidimensional system and system processing, Vol 12, 2001, pp199-208.
- M. Wang, C.P.Kwong, Computing GCLF using Syzygy algorithm, Proceedings of ICMS, 2002.

Research papers

- M.Wang,D.Feng, On Lin-Bose problem, Linear algebra and application, vol. 290, 2004;
- M. Wang, C.P.Kwong, On multivariate matrix factorization problems. Mathematics of Control, signals and systems. 2005.

Zero prime factorization

A full row rank matrix is **zero prime** if all its maximal order minors generate the unit ideal.

Lin-Bose propose the following problem:

Given F, full row rank of size lxm, if all the reduced minors generate the unit ideal, then whether or not F can be factorized as F=GxF1, with det(G)=d, d the gcd, and F1 being zero prime?

Zero prime factorization

J.F.Pommaret gave a proof using algebraic analysis method in 2001 Euro control conference, his method only holds for the **complex** number field.

We give a full proof for **any field**.
Linear algebra and applications, 2004

Minor prime factorization

- Let F be as above, d the gcd of all the l-minors of F. Whether or not there exsits G such that F=GxF1 and det(G)=d?
- If such factorization exists, we call it **minor prime factorization** of F.
- Several people gave counter-examples to showed that the above problem may have no solution in 1976-1979;
- However, **no one** propose a sufficient and necessary condition!

Minor prime factorization

We first found a **sufficient and necessary** condition.

Let K be submodule generated by all the rows of F. There exists a minor prime factorization if and only if the colon submodule K:d is a free module of rank l.

Remarks to the proof

The proof of the above result relies on a characterization of so-called **minor left prime**(MLP) matrix.

A full row rank matrix F of size lxm is said to be a MLP matrix if all the l-minors have only trivial common divisors.

The result just holds for any field. See *Mathematics of control, signals and systems, 2005* for details.

Remarks to proof

We first prove, a matrix F is a MLP if and only if K:d=K, where K=submodule generated by all the rows of F, d the g.c.d of all the lxl minors of F.

Then using a lifting to the linear mapping, we get the factor.

Note that when F is not full rank, this result will not hold, and above result does not hold either.

An algorithm

Based on above theorem, we propos an algorithm as follows:

Step 1 Let F as above, d the g.c.d of all l times l minors of F . Compute the syzygy module of row vectors of F and –dxI_m. If this syzygy module is free of rank l, then form a generating matrix [G|H] such that G is a lxl matrix, thus dG^{-1} is the desired matrix factor;

Step 2 If this syzygy module is not free, then there do not exist such a matrix factor.

Continue

We can easily check if a submodule is free using Fitting ideals.

Factor prime matrix

F is said to be factor prime, if for any square matrix G, F=GxF1, implies that det(G) is a non-zero constant.

A basic problem is to check if F is factor prime matrix .

Factor prime matrix

Checking if a matrix is factor prime is a long-standing open problem since the concept was proposed in 1979.

D.C.Youla, G,Gnavi, Notes on n-Dimensional systems theory, IEEE circuits and systems, vol. CAS-26, 1979.

We have also obtained some partial answers!

Remarks for checking of FLP

- No any other methods available to attack deciding problem of factor prime matrix;
- This is the first result concerning factor prime matrix.
- Full solutions can not obtained at this time, to be submit.

Remarks to the algorithm

Major problem is : how to extract a system of generators with l elements from a system of generators of syzygy module?

Book "Computational Commutative algebra I"by Martin Kreuzer and Lorenzo Robbiano provides a partial answer in Corollary 3.1.12 which was implemented in CoCoA.

Compare with other related work

- Previous papers only deal with very special cases.
- Above results are true for any field, that is, without restriction to characteristic zero and algebraically closed.
- Only Groebner bases theory is available to attack matrix factorization problems currently.

Some problems

I'd like to list some problems which deserve to be considered :

- Problem 1: if there are other efficient methods to deal with this problem?
- Problem 2: How to find a minimal generating system from a given generating system?
- Problem 3: If there are other criteria to assure that the existence of a generating system of l elements? (l=rank)

Some problems

- Problem 4: If elementary transformations for multivariate polynomial matrices can be used to find matrix fator?
- Problem 5: Find other characterizations for the existence of the matrix factor!
- Problem 6: Find the methods for dealing with non-full rank cases.

Conclusion

- Matrix factorization problems connects with many problems in system theory, for example, rational matrix fraction description.
- Generalization of convolutional codes in one variable to multidimension also needs to develop multivariate matrix theory.
- It needs to develop new algorithmic methods!

Thank you for your attention!